ON THE CONNECTEDNESS OF THE BRANCH LOCUS OF RATIONAL MAPS

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Abstract. Milnor proved that the moduli space M_d of rational maps of degree $d \geq 2$ has a complex orbifold structure of dimension $2(d-1)$. Let us denote by $\mathcal{S}_d$ the singular locus of M_d and by $\mathcal{B}_d$ the branch locus, that is, the equivalence classes of rational maps with non-trivial holomorphic automorphisms. Milnor observed that we may identify M_2 with $\mathbb{C}^2$ and, within that identification, that $\mathcal{B}_2$ is a cubic curve; so $\mathcal{B}_2$ is connected and $\mathcal{S}_2 = \emptyset$. If $d \geq 3$, then $\mathcal{S}_d = \mathcal{B}_d$. We use simple arguments to prove the connectivity of it.

1. Introduction

The space Rat_d of complex rational maps of degree $d \geq 2$ can be identified with a Zariski open set of the $(2d+1)$-dimensional complex projective space $\mathbb{P}^{2d+1}_{\mathbb{C}}$; this is the complement of the algebraic hypersurface defined by the resultant of two polynomials of degree at most d.

The group of Möbius transformations $\mathrm{PSL}_2(\mathbb{C})$ acts on Rat_d by conjugation: $\phi, \psi \in \mathrm{Rat}_d$ are said to be equivalent if there is some $T \in \mathrm{PSL}_2(\mathbb{C})$ so that $\psi = T \circ \phi \circ T^{-1}$. The $\mathrm{PSL}_2(\mathbb{C})$-stabilizer of $\phi \in \mathrm{Rat}_d$, denoted as $\mathrm{Aut}(\phi)$, is the group of holomorphic automorphisms of ϕ. As the subgroups of $\mathrm{PSL}_2(\mathbb{C})$ keeping invariant a finite set of cardinality at least 3 must be finite, it follows that $\mathrm{Aut}(\phi)$ is finite. Levy [5] observed that the order of $\mathrm{Aut}(\phi)$ is bounded above by a constant depending on d.

The quotient space $\mathrm{M}_d = \mathrm{Rat}_d/\mathrm{PSL}_2(\mathbb{C})$ is the moduli space of rational maps of degree d. Silverman [8] obtained that M_d carries the structure of an affine geometric quotient, Milnor [7] proved that it also carries the structure of a complex orbifold of dimension $2(d-1)$ (Milnor also obtained that $\mathrm{M}_2 \cong \mathbb{C}^2$) and Levy [5] noted that M_d is a rational variety. Let us denote by $\mathcal{S}_d \subset \mathrm{M}_d$ the singular locus of M_d, that is, the set of points over which M_d fails to be a topological manifold. The branch locus of M_d is the set $\mathcal{B}_d \subset \mathrm{M}_d$ consisting of those (classes of) rational maps with non-trivial group of holomorphic automorphisms.

As $\mathrm{M}_2 \cong \mathbb{C}^2$, clearly $\mathcal{S}_2 = \emptyset$. Using this identification, the locus $\mathcal{B}_2$ corresponds to the cubic curve [4]

$$2x^3 + x^2y - x^2 - 4y^2 - 8xy + 12x + 12y - 36 = 0,$$

where the cuspid $(-6, 12)$ corresponds to the class of a rational map $\phi(z) = 1/z^2$ with $\mathrm{Aut}(\phi) \cong D_3$ (dihedral group of order 6) and all other points in the cubic corresponds to those classes of rational maps with the cyclic group C_2 as full group of holomorphic automorphisms. In this way, $\mathcal{B}_2$ is connected.

If $d \geq 3$, then $\mathcal{S}_d = \mathcal{B}_d$ [6]. In this paper we observe the connectivity of $\mathcal{S}_d$.

Theorem 1. *If $d \geq 3$, then the singular locus $\mathcal{S}_d = \mathcal{B}_d$ is connected.*

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The proof of Theorem 1 will be a consequence of the description of the loci of classes of rationals maps admitting a given cyclic group of holomorphic automorphisms. For it, in Section 2, we recall known results concerning rational maps with non-trivial holomorphic automorphisms (see for instance [3, 6]). Then, we observe that the loci in moduli space consisting of classes of rational maps admitting the cyclic group C_n, $n \geq 2$, as group of holomorphic automorphisms is connected and we prove that, for $n \geq 3$ prime, such a locus always intersect the locus corresponding to C_2.

Theorem 1 states that given any two rational maps $\phi, \psi \in \mathrm{Rat}_d$, both with non-trivial group of holomorphic automorphisms, there is some $\rho \in \mathrm{Rat}_d$ which is equivalen to ψ and there is a continuous family $\Theta : [0,1] \to \mathrm{Rat}_d$ with $\Theta(0) = \phi$, $\Phi(1) = \rho$ and $\mathrm{Aut}(\Theta(t))$ non-trivial for every t. At this point we need to observe that if $\mathrm{Aut}(\phi) \cong \mathrm{Aut}(\psi)$, we may not ensure that $\mathrm{Aut}(\Theta(t))$ stay in the same isomorphic class; this comes from the existence of rigid rational maps [3] (in the non-cyclic situation).

Remark 1. *In the* 80*'s Sullivan provided a dictionary between dynamic of rational maps and the dynamic of Kleinian groups* [9]. *If we restrict to Klenian groups being co-compact Fuchsian groups of a fixed genus $g \geq 2$, then we are dealing with closed Riemann surfaces of genus g whose moduli space $\mathcal{M}_g$ has the structure of an orbifold of complex dimension* $3(g-1)$. *The branch locus in $\mathcal{M}_g$ is the set of isomorphic classes of Riemann surfaces with non-trivial holomorphic automorphisms. In* [1] *it was proved that in general the branch locus is non-connected; a difference with the connectivity of brach locus for rational maps.*

2. Rational maps with non-trivial group of holomorphic automorphisms

It is well known that a non-trivial finite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is either isomorphic to a cyclic group C_n or the dihedral group D_n or one of the alternating groups $\mathcal{A}_4, \mathcal{A}_5$ or the symmetric group $\mathfrak{S}_4$ (see, for instance, [2]). So, the group of holomorphic automorphisms of a rational map of degree at least two is isomorphic to one of the previous ones. Moreover, for each such finite subgroup there is a rational map admitting it as group of holomorphic automorphisms [3].

Let G be either C_n ($n \geq 2$), D_n ($n \geq 2$), $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathfrak{S}_4$. Let us denote by $\mathcal{B}_d(G) \subset \mathrm{M}_d$ the locus of classes of rational maps ϕ with $\mathrm{Aut}(\phi)$ containing a subgroup isomorphic to G. We say that G is admissible for d if $\mathcal{B}_d(G) \neq \emptyset$.

If G is either C_n or D_n or $\mathcal{A}_4$, then there may be some elements in $\mathcal{B}_d(G)$ with full group of holomorphic automorphisms non-isomorphic to G (i.e., they admit more holomorphic automorphisms than G). If G is either isomorphic to $\mathfrak{S}_4$ or $\mathcal{A}_5$, then every element in $\mathcal{B}_d(G)$ has G as its full group of holomorphic automorphisms and it may have isolated points [3], so it is not connected in general.

Below we recall a description of those values of d for which G is admissible (the results we present here are described in [6]) and we compute the dimension of $\mathcal{B}_d(G)$. Our main interest will be in the cyclic case, in which case we present the explicit computations, but we recall the general situation as a matter of completeness.

2.1. **Admissibility in the cyclic case.** In the case $G = C_n$, $n \geq 2$, the admissibility will depend on d. First, let us observe that if a rational map admits C_n as a group of holomorphic automorphisms, then we may conjugate it by a suitable Möbius transformation so that we may assume C_n to be generated by the rotation $T(z) = \omega_n z$, where $\omega_n = e^{2\pi i/n}$.

Theorem 2. *Let $d, n \geq 2$ be integers. The group C_n is admissible for d if and only if d is congruent to either $-1, 0, 1$ modulo n. Moreover, for such values, every rational map of degree d admitting C_n as a group of holomorphic automorphisms is equivalent to one of the form $\phi(z) = z\psi(z^n)$, where*

$$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \mathrm{Rat}_r,$$

satisfies that

(a) $a_r b_0 \neq 0$, *if* $d = nr + 1$*;*
(b) $a_r \neq 0$ *and* $b_0 = 0$, *if* $d = nr$*;*
(c) $a_r = b_0 = 0$ *and* $b_r \neq 0$, *if* $d = nr - 1$.

In the bove case, C_n is generated by the rotation $T(z) = \omega_n z$.

Proof. Let ϕ be a rational map admitting a holomorphic automorphism of order n. By conjugating it by a suitable Möbius transformation, we may assume that such automorphism is the rotation $T(z) = \omega_n z$.

(1) Let us write $\phi(z) = z\rho(z)$. The equality $T \circ \phi \circ T^{-1} = \phi$ is equivalent to $\rho(\omega_n z) = \rho(z)$. Let

$$\rho(z) = \frac{U(z)}{V(z)} = \frac{\sum_{k=0}^{l} \alpha_k z^k}{\sum_{k=0}^{l} \beta_k z^k},$$

where either $\alpha_l \neq 0$ or $\beta_l \neq 0$ and $(U, V) = 1$.

The equality $\rho(\omega_n z) = \rho(z)$ is equivalent to the existence of some $\lambda \neq 0$ so that

$$\omega_n^k \alpha_k = \lambda\alpha_k, \quad \omega_n^k \beta_k = \lambda\beta_k.$$

By taking $k = l$, we obtain that $\lambda = \omega_n^l$. So the above is equivalent to have, for $k < l$,

$$\omega_n^{l-k} \alpha_k = \alpha_k, \quad \omega_n^{l-k} \beta_k = \beta_k.$$

So, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $l - k \equiv 0 \mod (n)$. As $(U, V) = 1$, either $\alpha_0 \neq 0$ or $\beta_0 \neq 0$; so $l \equiv 0 \mod (n)$. In this way, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $k \equiv 0 \mod (n)$. In this way, $\rho(z) = \psi(z^n)$ for a suitable rational map $\psi(z)$.

(2) It follows from (1) that $\phi(z) = z\psi(z^n)$, for $\psi \in \mathrm{Rat}_r$ and suitable r. We next provide relations between d and r. Let us write

$$\psi(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k},$$

where $(P, Q) = 1$ and either $a_r \neq 0$ or $b_r \neq 0$. In this way,

$$\phi(z) = \frac{zP(z^n)}{Q(z^n)} = \frac{z\sum_{k=0}^{r} a_k z^{kn}}{\sum_{k=0}^{r} b_k z^{kn}}.$$

Let us first assume that $Q(0) \neq 0$, equivalently, $\psi(0) \neq \infty$. Then $\phi(0) = 0$ and the polinomials $zP(z^n)$ and $Q(z^n)$ are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) \neq 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = 1 + nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr$.

Let us now assume that $Q(0) = 0$, equivalently, $\psi(0) = \infty$. Let us write $Q(u) = u^l \widehat{Q}(u)$, where $l \geq 1$ and $\widehat{Q}(0) \neq 0$; so $\deg(Q) = l + \deg(\widehat{Q})$. In this case,

$$\phi(z) = \frac{P(z^n)}{z^{ln-1}\widehat{Q}(z^n)}$$

and the polinomials $P(z^n)$ (of degree $n\deg(P)$) and $z^{ln-1}\widehat{Q}(z^n)$ (of degree $n\deg(Q)-1$) are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) \neq 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr - 1$.

Summarizing all the above, we have the following situations:

(i) If $\phi(0) = 0$ and $\phi(\infty) = \infty$, then $\psi(0) \neq \infty$ and $\psi(\infty) \neq 0$; in particular, $d = nr + 1$. This case corresponds to have $a_r b_0 \neq 0$.

(ii) If $\phi(0) = \infty = \phi(\infty)$, then $\psi(0) = \infty$ and $\psi(\infty) \neq 0$; in which case $d = nr$. This case corresponds to have $a_r \neq 0$ and $b_0 = 0$.

(iii) If $\phi(0) = 0 = \phi(\infty)$, then $\psi(0) \neq \infty$ and $\psi(\infty) = 0$; in particular, $d = nr$. This case corresponds to have $a_r = 0$ and $b_0 \neq 0$. But in this case, we may conjugate ϕ by $A(z) = 1/z$ (which normalizes $\langle T \rangle$) in order to be in case (ii) above.

(iv) If $\phi(0) = \infty$ and $\phi(\infty) = 0$, then $\psi(0) = \infty$ and $\psi(\infty) = 0$; in particular, $d = nr - 1$. This case corresponds to have $a_r = b_0 = 0$ (in which case $b_r \neq 0$ as ψ has degree r).

□

Corollary 1. *C_2 is admissible for every $d \geq 2$.*

The explicit description provided in Theorem 2 permits to obtain the connectivity of $\mathcal{B}_d(C_n)$ and its dimension.

Corollary 2. *If $n \geq 2$ and C_n is admissible for d, then $\mathcal{B}_d(C_n)$ is connected and*

$$\dim_{\mathbb{C}}(\mathcal{B}_d(C_n)) = \begin{cases} 2(d-1)/n, & d \equiv 1 \mod n \\ (2d-n)/n, & d \equiv 0 \mod n \\ 2(d+1-n)/n, & d \equiv -1 \mod n \end{cases}$$

Proof. (1) By Theorem 2, the rational maps in Rat_d admitting a holomorphic automorphism of order $n \geq 2$ are conjugated those of the form $\phi(z) = z\psi(z^n) \in \mathrm{Rat}_d$ for $\psi \in \mathrm{Rat}_r$ as described in the same theorem.

Let us denote by $\mathrm{Rat}_d(n, r)$ the subset of Rat_d formed by all those rational maps of the $\phi(z) = z\psi(z^n)$, where ψ satisfies the conditions in Theorem 2.

If $d = nr + 1$, then we may identify $\mathrm{Rat}_d(n, r)$ with an open Zariski subset of Rat_r; if $d = nr$, then it is identified with an open Zariski subset of a linear hypersurface of Rat_r; and if $d = nr - 1$, then it is identified with an open Zariski subspace of a linear subspace of codimension two of Rat_r. In each case, we have that $\mathrm{Rat}_d(n, r)$ is connected. As the projection of $\mathrm{Rat}_d(n, r)$ to M_d is exactly $\mathcal{B}_d(C_n)$, we obtain its connectivity.

(2) The dimension counting. We may see that, if $d = nr + 1$, then ψ depends on $2r + 1$ complex parameters; if $d = nr$, then ψ depends on $2r$ complex parameters; and if $d = nr - 1$, then ψ depends on $2r - 1$ complex parameters. The normalizer in $\mathrm{PSL}_2(\mathbb{C})$ of $\langle T \rangle$ is the 1-complex dimensional group $N_n = \langle A_\lambda(z) = \lambda z, B(z) = 1/z : \lambda \in \mathbb{C} - \{0\} \rangle$. If $U \in N_n$, then $U \circ \phi \circ U^{-1}$ will also have T as a holomorphic automorphism. In fact,

$$A_\lambda \circ \phi \circ A_\lambda^{-1}(z) = z\psi(z^n/\lambda^n),$$
$$B \circ \phi \circ B(z) = z/\psi(1/z^n).$$

In this way, there is an action of N_n over Rat_r so that the orbit of $\psi(u)$ is given by the rational maps $\psi(u/t)$, where $t \in \mathbb{C} - \{0\}$, and $1/\psi(1/u)$. In this way, we obtain the desired dimensions. □

2.2. **Admissibility in the dihedral case.** Let us now assume $\phi \in \mathrm{Rat}_d$ admits the dihedral group D_n, $n \geq 2$, as a group of holomorphic automorphisms. Up to conjugation, we may assume that D_n is generated by $T(z) = \omega_n z$ and $A(z) = 1/z$. By Theorem 2, we may assume that $\phi(z) = z\psi(z^n)$, where

$$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \mathrm{Rat}_r,$$

where either

(a) $a_r b_0 \neq 0$, if $d = nr + 1$;
(b) $a_r \neq 0$ and $b_0 = 0$, if $d = nr$;
(c) $a_r = b_0 = 0$ and $b_r \neq 0$, if $d = nr - 1$;

with the extra condition that $\psi(z) = 1/\psi(1/z)$. This last condition is equivalent to the existence of some $\lambda \neq 0$ so that

$$\lambda a_k = b_{r-k}, \quad \lambda b_k = a_{r-k}, \quad k = 0, 1, ..., r.$$

The above is equivalent to have $\lambda \in \{\pm 1\}$ and $b_k = \lambda a_{r-k}$, for $k = 0, 1, ..., r$. In particular, this asserts that $a_r = 0$ if and only if $b_0 = 0$ (so case (b) above does not hold). Also, as the normalizer of the dihedral group $D_n = \langle T(z) = \omega_n z, A(z) = 1/z \rangle$ is a finite group, the dimension of $\mathcal{B}_d(D_n)$ is the same as half the projective dimension of those rational maps ψ satisfying (a) or (c). So, we may conclude the following result.

Theorem 3. *Let $d, n \geq 2$ be integers. The dihedral group D_n is admissible for d if and only if d is congruent to either ± 1 modulo n. Moreover, for such values, every rational map of degree d admitting D_n as a group of holomorphic automorphisms is equivalent to one of the form $\phi(z) = z\psi(z^n)$, where*

$$\psi(z) = \pm \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} a_{r-k} z^k} \in \mathrm{Rat}_r,$$

satisfies that

(i) $a_r \neq 0$, *if* $d = nr + 1$;
(ii) $a_r = 0$ *and* $a_0 \neq 0$, *if* $d = nr - 1$.

In the above case, D_n is generated by the rotation $T(z) = \omega_n z$ and the involution $A(z) = 1/z$.
If $n \geq 2$ and D_n is admissible for d, then

$$\dim_{\mathbb{C}}(\mathcal{B}_d(D_n)) = \begin{cases} (d-1)/n, & d \equiv 1 \mod n \\ (d+1-n)/n, & d \equiv -1 \mod n \end{cases}$$

Remark 2. *(a) If we are in case (i) and "+" sign for ψ, then ϕ fixes both fixed points of T and both fixed points of A. But, if we are in case (i) and "-" sign for ψ, then ϕ fixes both fixed points of T and permutes both fixed points of A.*

(b) If we are in case (ii) and "+" sign for ψ, then ϕ permutes both fixed points of T and fixes both fixed points of A. But, if we are in case (ii) and "-" sign for ψ, then ϕ permutes both fixed points of T and also both fixed points of A.

(c) If $n \geq 3$, then cases (i) and (ii) cannot happen simultaneously. Also, in either case, we obtain that $\mathcal{B}_d(D_n)$ has two connected components (they correspond to the choices of the sign "+" or "-").

2.3. **Admissibility of the platonic cases.** Let us now assume that $\phi \in \mathrm{Rat}_d$ admits as group of holomorphic automorphisms either $\mathcal{A}_4$, $\mathcal{A}_5$ or $\mathfrak{S}_4$. We may assume, up to conjugation, that (see, for instance, [2])

(1) $\langle T_3, B : T_3^3 = B^2 = (T_3 \circ A)^3 = I\rangle \cong \mathcal{A}_4$;
(2) $\langle T_4, C : T_4^4 = C^2 = (T_4 \circ C)^3 = I\rangle \cong \mathfrak{S}_4$.
(3) $\langle T_5, D : T_5^5 = D^2 = (T_5 \circ D)^3 = I\rangle \cong \mathcal{A}_5$;

where

$$T_n(z) = \omega_n z, \quad \omega_n = e^{2\pi i/n},$$
$$A(z) = 1/z,$$
$$B(z) = \frac{(\sqrt{3}-1)\left(z+(\sqrt{3}-1)\right)}{2z-(\sqrt{3}-1)},$$
$$C(z) = \frac{(\sqrt{2}+1)\left(-z+(\sqrt{2}+1)\right)}{z+(\sqrt{2}+1)},$$
$$D(z) = \frac{\left(1+\sqrt{2-\omega_5-\omega_5^4}\right)\left(-z+\left(1+\sqrt{2-\omega_5-\omega_5^4}\right)\right)}{(1-\omega_5-\omega_5^4)z+\left(1+\sqrt{2-\omega_5-\omega_5^4}\right)}.$$

Working in a similar fashion as done for the dihedral situation, one may obtains the following.

Theorem 4 ([6])**.** *Let $d \geq 2$.*

(1) *$\mathcal{A}_4$ is admissible for d if and only if d is odd.*
(2) *$\mathcal{A}_5$ is admissible for d if and only if d is congruent modulo 30 to either 1, 11, 19, 21.*
(3) *$\mathfrak{S}_4$ is admissible for d if and only if d is coprime to 6.*

3. Proof of Theorem 1

It is clear that $\mathcal{B}_d$ is equal to the union of all $\mathcal{B}_d(G)$, where G runs over the admissible finite groups for d.

If G is admissible for d and p is a prime integer dividing the order of G (so that the cyclic group C_p is a subgroup of G), then C_p is admissible for d and $\mathcal{B}_d(G) \subset \mathcal{B}_d(C_p)$. In this way, $\mathcal{B}_d$ is equal to the union of all $\mathcal{B}_d(C_p)$, where p runs over all integer primes with C_p admissible for d. Corollary 2 asserts that each $\mathcal{B}_d(C_p)$ is connected. Now, the connectivity of $\mathcal{B}_d$ will be consequence of Lemma 1 below.

Lemma 1. *If $p \geq 3$ is a prime and C_p is admissible for d, then $\mathcal{B}_d(C_p) \cap \mathcal{B}_d(C_2) \neq \emptyset$.*

Proof. We only need to check the existence of a rational map $\phi \in \mathrm{Rat}_d$ admitting a holomorphic automorphism of order p and also a holomorphic automorphism of order 2.

First, let us consider those rational maps of the form $\phi(z) = z\psi(z^p)$, where (by Theorem 2) we may assume to be of the form

$$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \mathrm{Rat}_r,$$

with

(a) $a_r b_0 \neq 0$, if $d = pr + 1$;
(b) $a_r \neq 0$ and $b_0 = 0$, if $d = pr$;
(c) $a_r = b_0 = 0$, if $d = pr - 1$.

Assume we are in either case (a) or (c). By considering $b_k = a_{r-k}$, for every $k = 0, 1, ..., r$, we see that ψ satisfies the relation $\psi(1/z) = 1/\psi(z)$; so ϕ also admits the holomorphic automorphism $A(z) = 1/z$. The automorphisms $T(z) = \omega_p z$ and A generate a dihedral group of order $2p$.

In case (b), we can consider ψ so that $\psi(-z) = \psi(z)$, which is posible to find if we assume that $(-1)^k a_k = (-1)^r a_k$ and $(-1)^k b_k = (-1)^r b_k$ (which means that $a_k = b_k = 0$ if k and r have different parity). In this case T and $V(z) = -z$ are holomorphic automorphisms of ϕ, generating the cyclic group of order $2p$.

□

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